of which $=$ *one*), r the radius of the circumscribing circle, and β the angle subtended by one of the equal sides,

$$x = \frac{r^2-1}{r^2}, \text{ and } r = \frac{1}{\sqrt{2\,(1-\cos\beta)}}.$$

SOLUTION OF TWO INDETERMINATE PROBLEMS.

BY GEO. R. PERKINS, LL. D., UTICA, N. Y.

PROBLEM I.—Find n numbers in arithmetical progression, such that the sum of their cubes shall be a square number.

Assume the n terms of the progression as follows:

$$x-\frac{n-1}{2}d;\ x-\frac{n-3}{2}d;\ \ldots . x-\tfrac{1}{2}d;\ x+\tfrac{1}{2}d;\ \ldots . x+\frac{n-3}{2}d;\ x+\frac{n-1}{2}d,$$

which corresponds to the case when n is an even number, and

$$x-\frac{n-1}{2}d;\ x-\frac{n-3}{2}d;\ \ldots . x-d;\ x;\ x+d;\ \ldots . x+\frac{n-3}{2}d;\ x+\frac{n-1}{2}d,$$

which corresponds to the case when n is an odd number.

The sum of the cubes of the n terms, when n is even, is

$$n\,x^3 + \tfrac{3}{2}\,[1^2 + 3^2 + 5^2 + \ldots . (n-3)^2 + (n-1)^2]\,d^2\,x.$$

When n is odd, the sum of the cubes is

$$n\,x^3 + 6\,[1^2 + 2^2 + 3^2 + \ldots . (\tfrac{1}{2}\,\overline{n-3})^2 + (\tfrac{1}{2}\,\overline{n-1})^2]d^2\,x.$$

Each of these expressions, when simplified, becomes

$$n\,x\,[x^2 + (n^2-1)\,(\tfrac{1}{2}\,d)^2] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and this must be a square number.

$$\text{Assume } x^2 + (n^2-1)\,(\tfrac{1}{2}\,d)^2 = 4\,n\,x\,t^2,$$

and the above will become $4\,n^2\,x^2\,t^2 =$ a square.

Solving $x^2 + (n^2-1)\,(\tfrac{1}{2}\,d)^2 = 4\,n\,x\,t^2$

for x, we find

$$x = 2\,n\,t^2 + \sqrt{4\,n^2\,t^4-(n^2-1)\,(\tfrac{1}{2}\,d)^2} \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Hence, in order that this value of x may be rational, we must make

$$4\,n^2\,t^4 - (n^2-1)\,(\tfrac{1}{2}\,d)^2 = \text{a square} = [2\,n\,t^2 - s\,(\tfrac{1}{2}\,d)]^2$$

$$\text{This gives } d = \frac{8\,n\,s\,t^2}{s^2+n^2-1}, \text{ and then}$$

$$x = \frac{4\,n\,(n^2-1)\,t^2}{s^2+n^2-1}.$$

The first term $= x - \frac{n-1}{2} d = \frac{4\,n\,(n-1)\,(n+1-s)\,t^2}{s^2 + n^2 - 1}$.

The last term $= x + \frac{n-1}{2} d = \frac{4\,n\,(n-1)\,(n+1+s)\,t^2}{s^2 + n^2 - 1}$.

And the sum of the cubes of all the terms

$$= 4\,n^2\,x^2\,t^2 = \frac{64\,n^4\,(n^2-1)^2\,t^6}{(s^2+n^2-1)^2}.$$

If we take $s=1$ and $t=\frac{1}{2}\,n\,r$, the foregoing expressions will be integral, as follows:

$$\left.\begin{array}{ll} & d = 2\,n\,r^2 \\ \text{The first term} & = n^2\,(n-1)\,r^2 \\ \text{The last term} & = n\,(n-1)\,(n+2)\,r^2 \\ \text{The sum of the cubes} & = n^6\,(n^2-1)^2\,r^6 \end{array}\right\} \cdots\cdots (c)$$

Again, if we wish our terms to correspond to the series of natural numbers, we assume, in (c), as follows:

$$r = \frac{1}{2\,m} \text{ and } n = 2\,m^2 = \text{number of terms,}$$

and we have

$$\left.\begin{array}{ll} & d = 1 \\ \text{The first term} & = m^2\;(2\,m^2-1) \\ \text{The last term} & = (m^2+1)\,(2\,m^2-1) \\ \text{Sum of the cubes} & = m^6\,(4\,m^4-1)^2 \end{array}\right\} \cdots\cdots (d)$$

As particular examples of formula (c), take $r = 1$ and $n =$ number of terms $= 3$, 4 and 5, in succession, and we find,

$$(18)^3 + (24)^3 + (30)^3 = (216)^2,$$
$$(48)^3 + (56)^3 + (64)^3 + (72)^3 = (960)^2,$$
$$(100)^3 + (110)^3 + (120)^3 + (130)^3 + (140)^3 = (3000)^2.$$

As particular examples of formula (d), take $m = 1$, 2, 3 and 4, in succession, and we find,

$$1^3 + 2^3 = 3^2,$$
$$(28)^3+(29)^3+(30)^3+(31)^3+(32)^3+(33)^3+(34)^3+(35)^3 = (504)^2,$$
$$(153)^3+(154)^3+\ldots\ldots(169)^3+(170)^3 = (8721)^2,$$
$$(496)^3+(497)^3+\ldots\ldots(526)^3+(527)^3 = (65472)^2.$$

Remark.—A solution of this Problem, when $d = 1$, is given by M. Eugène Catalan in the XX.th volume of the Acts of the Accademia Pontificia Dè Nouvi Lincei, Rome, 1866.

This solution is very satistactory, but very lengthy, involving very large numbers, and is, moreover, not a general solution.

His numbers are

$$[887240758600]^3 + [887240758601]^3 + \ldots\ldots + [1041543499225]^3$$
$$= [967473261775 \times 77151370313]^2.$$

PROBLEM II.—Find n consecutive terms, in the natural series of numbers, such that the sum of their cubes shall be a cube number.

Using the same notation as in the last problem, and making $d = 1$, our expression (a) for the sum of the cubes becomes

$$n\,x\left[x^2 + \tfrac{1}{4}(n^2 - 1)\right],$$

and this must be a cube, which is obviously the case, when $x = \frac{1}{2}$.

We will therefore assume $x = \frac{1}{2} + r$, and our expression will become

$$n\left\{\frac{n^2}{8} + \frac{n^2+2}{4}r + \frac{3}{2}r^2 + r^3\right\}.$$

Put $n = p^3 =$ number of terms, and this will become

$$p^3\left\{\frac{p^6}{8} + \frac{p^6+2}{4}r + \frac{3}{2}r^2 + r^3\right\} = \text{cube} = p^3\left\{\frac{p^2}{2} + r\right\}^3.$$

Hence $r = \dfrac{p^4 - 2p^2 - 2}{6}$, and $x = \frac{1}{2} + r = \dfrac{p^4 - 2p^2 + 1}{6}$.

$$\left.\begin{array}{l} \text{First term} = x - \dfrac{p^3-1}{2} = \dfrac{p^4-3p^3-2p^2+4}{6}, \\ \text{Last term} = x + \dfrac{p^3-1}{2} = \dfrac{p^4+3p^3-2p^2-2}{6}, \\ \text{The sum of the cubes} = \left\{\dfrac{p(p-1)(p+1)(p^2+2)}{6}\right\}^3 \end{array}\right\} \ldots\ldots (e)$$

As particular cases, first suppose $p = 2$ and we find

$$\frac{p^4 - 3p^3 - 2p^2 + 4}{6} = -2, \text{ the first term.}$$

Hence the 8 terms denoted by p^3 will be $-2, -1, 0, 1, 2, 3, 4, 5$.

The cubes of the two negative values -1 and -2 will balance the cubes of the two corresponding positive values, so that the sum of the cubes of these 8 terms will be reduced to $3^3 + 4^3 + 5^3$. In this case

$$\frac{p(p-1)(p+1)(p^2+2)}{6} = 6.$$

Hence, we have this remarkable result. $3^3 + 4^3 + 5^3 = 6^3$, which might have been obtained from the conditions $(18)^3 + (24)^3 + (30)^3 = (216)^2$ of the first problem, by dividing by 6^3.

When $p = 4$, we find, $6^3 + 7^3 + 8^3 + \ldots + (68)^3 + (69)^3 = (180)^3$.

When $p = 5$, we have, $(34)^3\ (35)^3 + \ldots + (157)^3 + (158)^3 = (540)^3$.

When $p = 10$, that is, when the number of terms $= p^3 = 1000$, we have $(1134)^3 + (1135)^3 + \ldots + (2132)^3 + (2133)^3 = (16830)^3$.

REMARK —In the *Mathematical Diary* for 1831, on page 186, it is stated that M. Pagliani had published his solution to this problem, in this last case, when the number of terms is 1000, in the "Annales de Mathematiques" by M. Gergonne.

In the *Mathematical Miscellany* for 1839, on page 127, William Lenhart has given a general solution of this problem, and as a particular case has obtained the same 1000 terms, as were given by M. Pagliani, and I wish, here, to express my high estimation of Mr. Lenhart's valuable contributions to this particular department of mathematics, given in the pages of the *Mathematical Miscellany.*

PROBLEM RELATING TO THE DETERMINATION OF CIRCULAR ORBITS.

BY G. W. HILL, ESQ., NYACK TURNPIKE, N. Y.

Determine the elements of the orbit of a planet or satellite, which moves in a circle in the plane of the ecliptic, from three observations of its direction from the earth, made at equal intervals of time; the positions of the earth and the central body at these times being known, but the sum of the masses of the central body and the planet or satellite being unknown.

Or, geometrically stated,—

In a plane, given a point as center and three straight lines, required to describe a circle, so that the arcs intercepted by the lines taken in a determinate order may be equal.

SOLUTION.

Let generally R denote the sun's distance from the earth,
" " L its longitude,
" " r the constant radius vector of the planet,
" " χ its heliocentric longitude,
" " η its heliocentric angular motion from one observation to the next,
" " λ its longitude as seen from the earth,
" " $\triangle$ its distance from the earth.

Moreover, employ the subscribts $(_{-1})$, $(_0)$, $(_1)$, to denote the special values of the above quantities, which have place at the three times of observation in their order.

By the theory of the transformation of rectangular co-ordinates from